Modeling the White Light Fringe

Slava G. Turyshev

Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA 91109

ABSTRACT

This paper addresses the issue of modeling the white light fringe. We developed analytic technique for extracting the phase, visibility and amplitude information as needed for interferometric astrometry with the Space Interferometry Mission (SIM). The model accounts for a number of instrumental and physical effects and is able to compensate for a number of operational regimes. In particular, we were able to obtain general solution for polychromatic phasors and address properties of unbiased fringe estimators in the presence of noise. For demonstration purposes we studied the case of rectangular bandpass filter with two different methods of optical path difference (OPD) modulation – stepping and ramping OPD modulations. A number of areas of further studies relevant to instrument design and simulations are outlined and discussed.

Keywords: SIM, metrology, pathlength feedforward, astrometry, modeling

1. INTRODUCTION

SIM is designed as a space-based 10-m baseline Michelson optical interferometer operating in the visible waveband (see Ref.[1] for more details). This mission will open up many areas of astrophysics, via astrometry with unprecedented accuracy. Over a narrow field of view SIM is expected to achieve a mission accuracy of 1 μas. In this mode SIM will search for planetary companions to nearby stars by detecting the astrometric "wobble" relative to a nearby ($\leq 1^\circ$) reference star. In its wide-angle mode, SIM will be capable to provide a 4 μas precision absolute position measurements of stars, with parallaxes to comparable accuracy, at the end of a 5-year mission. The expected proper motion accuracy is around 3 μas/yr, corresponding to a transverse velocity of 10 m/s at a distance of 1 kpc.

The SIM instrument does not directly measure the angular separation between stars, but the projection of each star direction vector onto the interferometer baseline by measuring the pathlength delay of starlight as it passes through the two arms of the interferometer. The delay measurement is made by a combination of internal metrology measurements to determine the distance the starlight travels through each arm, external metrology measurements that determine the length and local orientation of the baseline, and a measurement of the central white light fringe to determine the point of equal optical pathlength. The current algorithms and simulations for optical interferometry are all based on monochromatic light. This is a good approximation for some of existing testbed configurations that use as many as 80 spectral channels for dispersed light. Nominally the flight system will use four to eight channels for guide interferometers. Because of the large bandwidth of each channel (87.5 nm), the quasi-monochromatic assumptions are not valid, and modifications to the algorithms are necessary.

This paper discusses analytic model for the white light fringe data extraction. Our goal here is to establish functional dependency of the white light fringe parameters on the instrumental input parameters. The problem of interference of electromagnetic radiation is well studied and extensive number of publications on this subject are available (see Refs.[2–10] and references therein). While numerical studies have proven to be extremely valuable in analyzing the interference patterns and are very useful in addressing various instrumental effects, the analytical methods may provide the much needed critical understanding of the white light interference phenomena. It will be demonstrated below that analytic solution may be used as a tool to study the complex interferometric phenomena on a principally different qualitative level.[2,7]

For correspondence E-mail: .turyshev@jpl.nasa.gov

2. PARAMETERIZATION OF POLYCHROMATIC FRINGE PATTERN

Description of the interferometric pattern in the polychromatic case that involves a finite bandwidth of radiation - is a complicated task. Thus, the observational conditions in the case of polychromatic light are significantly altered compare to the simplicity of the monochromatic process. In general, all the quantities involved are complicated functions of the wavelength. A way to describe this process is to collect contributions of all infinitesimal constituents of polychromatic light at different wavelengths within the bandwidth of the incoming electromagnetic radiation.[2] In other words, the total number of photo-electron counts, N, registered by a CCD detector per wavenumber and per unit time, may be given by the following expression:

$$dN(k,t) = \mathcal{F}(k)\mathcal{I}_0(k)\Big(1 + V(k)\sin\big[\phi(k) + kx(t)\big]\Big)dk\,dt, \tag{1}$$

where $\mathcal{F}(k)$ is a dimensionless factor representing the total instrumental throughput; $\mathcal{I}_0(k)$, $V(k)$ and $\phi(k)$ are the intensity, visibility and phase of the incoming light; $x(t)$ is modulated internal delay. We are using a nomenclature where a wavenumber k relates to the wavelength as follows $k = \frac{2\pi}{\lambda}$. We also accounted for the nominal $\frac{\pi}{2}$ phase shift due to the SIM beam splitter, which produces a sine fringe rather than a cosine one.

Note that the total instrumental throughput depends on a number of other factors, some of these are the collective area of the detector, quantum efficiency of CCD, and overall spectral response of the instrument. Our goal here is to derive observational equation that may be used to estimate the apparent fringe phase and visibility. Effects that are not included in the model are due to polarization of both incoming light and the instrumental throughput, effect of the wavefront-tilt, low frequency vibrations, drifts, jitter, etc. To estimate the true source visibility and phase one would have to perform a set of additional calibration and estimation procedures that will be addressed elsewhere.

2.1. Integration Over the Spectral Bandwidth

Let us first perform integration over the SIM wavenumber bandwidth $k \in [k^-_{\tt SIM}, k^+_{\tt SIM}]$, where $k^-_{\tt SIM} = 450$ nm is the beginning of the SIM bandwidth, and $k^+_{\tt SIM} = 950$ nm is the end of this bandwidth, thus $k \in [450, 950]$ nm. A formal integration of Eq.(1) over dk leads to the following result

$$N(t)\Delta k_{\tt SIM} \quad = \quad \int_{k^-_{\tt SIM}}^{k^+_{\tt SIM}} N(k,t)\,dk, \qquad\qquad \Delta k_{\tt SIM} = k^+_{\tt SIM} - k^-_{\tt SIM}. \tag{2}$$

In the case of channeled (or dispersed) spectrum output, the integration of this equation over the range of wavenumbers is straightforward. For this purpose, we designate index, ℓ, to denote a particular spectral channel. Suppose that there exists a total of L spectral channels, thus $\ell \in [1, L]$.

Definition for the spectral channel ℓ implies the width of the channel $\Delta k_\ell = k^+_\ell - k^-_\ell$ and existence of a "central" wavenumber k_ℓ within this channel. We also, assume continuous spectrum within the bandwidth, so that there is no gaps exist in the interval $k \in [k^-_{\tt SIM}, k^+_{\tt SIM}]$. A consequence of this is the equality $k^-_{\ell+1} = k^+_\ell$, which leads to the following discrete representation of the bandwidth

$$\Delta k_{\tt SIM} = k^+_{\tt SIM} - k^-_{\tt SIM} = \sum_{\ell=1}^{L} (k^+_\ell - k^-_\ell) \equiv \sum_{\ell=1}^{L} \Delta k_\ell. \tag{3}$$

Result Eq. (3) allows us to rewrite Eq. (2) as follows:

$$N(t)\Delta k_{\tt SIM} = \int_{k^-_{\tt SIM}}^{k^+_{\tt SIM}} N(k,t)\,dk = \sum_{\ell=1}^{L} \int_{k^-_\ell}^{k^+_\ell} N(k,t)\,dk = \sum_{\ell=1}^{L} N_\ell(t)\Delta k_\ell, \tag{4}$$

where $N_\ell(t)$ is the instantaneous number of photons within a particular spectral channel given as below

$$N_\ell(t) = \frac{1}{\Delta k_\ell}\int_{k^-_\ell}^{k^+_\ell} \mathcal{F}(k)\mathcal{I}_0(k)\Big(1 + V(k)\sin\big[\phi(k) + kx(t)\big]\Big)dk. \tag{5}$$

This equation will help to focus our attention from the discussion of coherent processes within the whole wide bandwidth, onto addressing these processes on a smaller scale – within a particular narrow spectral channel, ℓ.

2.2. Definitions for the Fringe Parameters

At this point, it is convenient to introduce a set of useful notations. First of all, we define the average total intensity of incoming electromagnetic radiation, $\mathcal{I}_{0\ell}$, within the ℓ-th spectral channel as

$$\mathcal{I}_{0\ell} = \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \mathcal{F}(k)\mathcal{I}_0(k)dk. \tag{6}$$

It is natural to introduce normalized intensity of light $\hat{\mathcal{I}}_{0\ell}$ within the ℓ-th channel:

$$\hat{\mathcal{I}}_{0\ell}(k) = \frac{\mathcal{F}(k)\mathcal{I}_0(k)}{\mathcal{I}_{0\ell}} \qquad \text{with} \qquad \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{\mathcal{I}}_{0\ell}(k)dk = 1. \tag{7}$$

These new notations allow to present Eq. (5) as given below

$$N_\ell(t) = \mathcal{I}_{0\ell}\Big(1 + \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{\mathcal{I}}_{0\ell}(k)V(k)\sin\big[\phi(k) + kx(t)\big]dk\Big). \tag{8}$$

2.2.1. Fringe Visibility, Mean Wavenumber and Phase

To further simplify the obtained equation, we will introduce functional form the fringe visibility, the phase and the wavenumber notations. Thus, the fringe visibility, $V_{0\ell}$, within the ℓ-th channel is given as

$$V_{0\ell} = \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{\mathcal{I}}_{0\ell}(k)V(k)dk. \tag{9}$$

Similarly to Eq. (7) we denote normalized visibility in the channel as

$$\hat{V}_{0\ell}(k) = \frac{\hat{\mathcal{I}}_{0\ell}(k)V(k)}{V_{0\ell}} \equiv \frac{\mathcal{F}(k)\mathcal{I}_0(k)V(k)}{\mathcal{I}_{0\ell}V_{0\ell}}, \qquad \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{V}_{0\ell}(k)dk = 1. \tag{10}$$

These definitions help us to re-write equation (8) in the following compact form

$$N_\ell(t) = \mathcal{I}_{0\ell}\Big(1 + V_{0\ell}\frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{V}_{0\ell}(k)\sin\big[\phi(k) + kx(t)\big]dk\Big). \tag{11}$$

We define mean wavenumber, k_ℓ, and mean phase, ϕ_ℓ, for the ℓ-th spectral channel as given below

$$k_\ell \;=\; \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{\mathcal{I}}_{0\ell}(k)\, k\, dk \;\equiv\; \frac{1}{\mathcal{I}_{0\ell}\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \mathcal{F}(k)\mathcal{I}_0(k)\, k\, dk, \tag{12}$$

There are two ways to define the phase within the channel. Thus, it is tempting to define the mean phase as

$$\phi_\ell = \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{\mathcal{I}}_{0\ell}(k)\phi(k)dk \;\equiv\; \frac{1}{\mathcal{I}_{0\ell}\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \mathcal{F}(k)\mathcal{I}_0(k)\phi(k)dk. \tag{13}$$

This is acceptable for narrow spectral channel, however for a wide channel one needs a more convenient form:

$$\phi(k_\ell), \qquad \text{which is} \qquad \phi(k_\ell) \neq \phi_\ell, \tag{14}$$

and is simply the phase value at the specific wavenumber. In our further analysis we will be using this later definition. The three introduced quantities (i.e. the visibility, mean wavenumber k_ℓ and phase at the mean wavenumber $\phi(k_\ell)$) allow to proceed with integration of Eq. (11).

2.3. Complex Fringe Envelope Function

Definitions introduced in the previous Section allow us to separate functions k_ℓ and $\phi(k_\ell)$ from the functions with dexplicit dependency on the wavenumber k. As a result, Eq. (11) may be presented as below

$$\begin{aligned} N_\ell(t) &= \mathcal{I}_{0\ell}\Big(1 + V_{0\ell}\sin\left[\phi(k_\ell)+k_\ell x(t)\right]\cdot\frac{1}{\Delta k_\ell}\int_{k_\ell^-}^{k_\ell^+}\hat{V}_{0\ell}(k)\cos\left[(k-k_\ell)x(t)+\phi(k)-\phi(k_\ell)\right]dk + \\ &\quad + V_{0\ell}\cos\left[\phi(k_\ell)+k_\ell x(t)\right]\cdot\frac{1}{\Delta k_\ell}\int_{k_\ell^-}^{k_\ell^+}\hat{V}_{0\ell}(k)\sin\left[(k-k_\ell)x(t)+\phi(k)-\phi(k_\ell)\right]dk\Big). \end{aligned} \tag{15}$$

To further simplify the analysis, we introduce the complex fringe envelope function, $\tilde{W}_\ell\left[\Delta k_\ell, \phi(k_\ell), x(t)\right]$:

$$\tilde{W}_\ell\left[\Delta k_\ell, \phi(k_\ell), x(t)\right] = \frac{1}{\Delta k_\ell}\int_{k_\ell^-}^{k_\ell^+}\hat{V}_{0\ell}(k)\, e^{j\left((k-k_\ell)x(t)+\phi(k)-\phi(k_\ell)\right)}dk. \tag{16}$$

As a complex function, $\tilde{W}_\ell$ may be equivalently presented by its real, $\mathsf{Re}\{\tilde{W}_\ell\}$, and imaginary, $\mathsf{Im}\{\tilde{W}_\ell\}$, components:

$$\tilde{W}_\ell = \mathsf{Re}\{\tilde{W}_\ell\} + j\ \mathsf{Im}\{\tilde{W}_\ell\}, \qquad \text{with} \tag{17}$$

$$\mathsf{Re}\{\tilde{W}_\ell\} = \frac{1}{\Delta k_\ell}\int_{k_\ell^-}^{k_\ell^+}\hat{V}_{0\ell}(k)\cos\left[(k-k_\ell)x(t)+\phi(k)-\phi(k_\ell)\right]dk,$$

$$\mathsf{Im}\{\tilde{W}_\ell\} = \frac{1}{\Delta k_\ell}\int_{k_\ell^-}^{k_\ell^+}\hat{V}_{0\ell}(k)\ \sin\left[(k-k_\ell)x(t)+\phi(k)-\phi(k_\ell)\right]dk. \tag{18}$$

The complex envelope function, given by Eq.(16)-(18), allows us to present expression (15) in a simpler form:

$$N_\ell(t) = \mathcal{I}_{0\ell}\Big(1 + V_{0\ell}\sin\left[\phi(k_\ell)+k_\ell x(t)\right]\cdot\mathsf{Re}\Big\{\tilde{W}_\ell\left[x(t)\right]\Big\} + V_{0\ell}\cos\left[\phi(k_\ell)+k_\ell x(t)\right]\cdot\mathsf{Im}\Big\{\tilde{W}_\ell\left[x(t)\right]\Big\}\Big). \tag{19}$$

Function, $\tilde{W}_\ell\left[x(t)\right]$, as any complex function, may also be represented by its amplitude and its phase, namely:

$$\tilde{W}_\ell\left[\Delta k_\ell, \phi(k_\ell), x(t)\right] = \mathcal{E}_\ell\left[\Delta k_\ell, \phi(k_\ell), x(t)\right] e^{i\Omega_\ell\left[\Delta k_\ell, \phi(k_\ell), x(t)\right]}, \tag{20}$$

where $\mathcal{E}_\ell$ and Ω_ℓ are the amplitude and phase correspondingly. For the complex envelope function Eq. (17) these two are given by $\mathcal{E}_\ell(t) = \sqrt{\mathsf{Re}^2\{\tilde{W}_\ell\} + \mathsf{Im}^2\{\tilde{W}_\ell\}}$, and $\Omega_\ell(t) = \mathsf{ArcTan}\Big\{\mathsf{Im}\{\tilde{W}_\ell\}/\mathsf{Re}\{\tilde{W}_\ell\}\Big\}$. Finally, we re-write Eq. (19) as

$$N_\ell(t) = \mathcal{I}_{0\ell}\Big(1 + V_{0\ell}\ \mathcal{E}_\ell(t)\cdot\sin\left[\phi(k_\ell)+k_\ell x(t)+\Omega_\ell(t)\right]\Big). \tag{21}$$

Note that the apparent visibility of the fringe now is the product of the true averaged visibility and the modulus of the phase corrected Fourier transform of the filter function, evaluated at the current delay or $\tilde{\Gamma}_x = \tilde{V}_{0\ell}\tilde{W}_\ell \equiv V_{0\ell}\,\mathcal{E}_\ell\, e^{j\left(\phi(k_\ell)+\Omega_\ell\right)}$. The transfer function $\tilde{W}_\ell$ describes the coherence envelope.[2] If $\hat{V}_{0\ell}(k) \sim \mathcal{F}(k)\mathcal{I}_0(k)V(k)$ is symmetric, then $\tilde{W}_\ell$ is real valued, $\Omega_\ell = 0$, and only at zero delay,[8] where the envelope is at peak, is the true visibility observed.

2.4. Temporal Integration

The last integration to be performed in Eq.(1) (or equivalently Eq.(21)), is the integration over time. The optical pathlength difference may be modulated either as a set of discrete values corresponding to a number of steps in the OPD space (stepping PZT modulation) or by ramping PZT over the range of OPD values. The total integration time, Δt, is the sum of durations of eight temporal bins.

$$\Delta t = t^+ - t^- = \sum_{i=1}^{N}\Delta\tau_i, \qquad \text{with} \qquad \Delta\tau_i = t_i^+ - t_i^-. \tag{22}$$

Direct integration of Eq. (19) leads to expression for the total number of photons collected at each PZT stroke:

$$N_\ell \, \Delta t \quad = \quad \int_{t^-}^{t^+} N_\ell(t)\,dt = \sum_{i=1}^{N} \int_{t_i^-}^{t_i^+} N_\ell(t)\,dt = \sum_{i=1}^{N} N_{\ell i}\Delta\tau_i. \tag{23}$$

where $N_{\ell i}\Delta\tau_i$ is the total number of photons collected in a particular i-th temporal bin and for the ℓ-th spectral channel. Substituting $N_\ell(t)$ from Eq. (21) directly into Eq. (23), one obtains following expression for $N_{\ell i}$:

$$N_{\ell i} \quad = \quad \frac{1}{\Delta\tau_i}\int_{t_i^-}^{t_i^+} dt\ \mathcal{I}_{0\ell}\Big(1 + V_{0\ell}\sin\phi(k_\ell)\cdot\mathcal{E}_\ell(t)\cdot\cos\big[k_\ell x(t)+\Omega_\ell(t)\big] + V_{0\ell}\cos\phi(k_\ell)\cdot\mathcal{E}_\ell(t)\cdot\sin\big[k_\ell x(t)+\Omega_\ell(t)\big]\Big). \tag{24}$$

To complete this integration, we assume that quantities $\mathcal{I}_{0\ell}$, $V_{0\ell}, \phi(k_\ell)$, and k_ℓ do not change with time during the photon-counting intervals. The only quantity that is explicitly varies with time – is the OPD $x(t)$.

Integration over time may be performed in a general form and expression for $N_{\ell i}$ is given as follows:

$$N_{\ell i} \quad = \quad \mathcal{I}_{0\ell}\Big(1 + V_{0\ell}\sin\phi(k_\ell)\cdot\mathsf{Re}\{\tilde{\mathcal{P}}_{\ell i}\} + V_{0\ell}\cos\phi(k_\ell)\cdot\mathsf{Im}\{\tilde{\mathcal{P}}_{\ell i}\}\Big), \tag{25}$$

with quantities $\mathsf{Re}\{\tilde{\mathcal{P}}_{\ell i}\}$ and $\mathsf{Im}\{\tilde{\mathcal{P}}_{\ell i}\}$ given as below:

$$\begin{aligned}
\mathsf{Re}\{\tilde{\mathcal{P}}_{\ell i}\} \quad &= \quad \frac{1}{\Delta\tau_i}\int_{t_i^-}^{t_i^+} dt\ \mathcal{E}_\ell\big[\Delta k_\ell,\phi(k_\ell),x(t)\big]\cos\big[k_\ell x(t)+\Omega_\ell[\Delta k_\ell,\phi(k_\ell),x(t)]\big], \\
\mathsf{Im}\{\tilde{\mathcal{P}}_{\ell i}\} \quad &= \quad \frac{1}{\Delta\tau_i}\int_{t_i^-}^{t_i^+} dt\ \mathcal{E}_\ell\big[\Delta k_\ell,\phi(k_\ell),x(t)\big]\sin\big[k_\ell x(t)+\Omega_\ell[\Delta k_\ell,\phi(k_\ell),x(t)]\big].
\end{aligned} \tag{26}$$

For convenience of further analysis, we combined these two real-valued matrices into one complex matrix $\tilde{\mathcal{P}}_{\ell i}$

$$\tilde{\mathcal{P}}_{\ell i} \quad = \quad \mathsf{Re}\{\tilde{\mathcal{P}}_{\ell i}\} + j\ \mathsf{Im}\{\tilde{\mathcal{P}}_{\ell i}\}. \tag{27}$$

Furthermore, using definitions Eq. (26), this complex matrix may be presented as given below

$$\tilde{\mathcal{P}}_{\ell i} \quad = \quad \frac{1}{\Delta\tau_i}\int_{t_i^-}^{t_i^+} dt\ \mathcal{E}_\ell e^{j\,\left(k_\ell x(t)+\Omega_\ell[\Delta k_\ell,\phi(k_\ell),x(t)]\right)} \equiv \frac{1}{\Delta\tau_i}\int_{t_i^-}^{t_i^+} dt\ e^{j\,k_\ell x(t)}\ \tilde{W}_\ell\big[\Delta k_\ell,\phi(k_\ell),x(t)\big], \tag{28}$$

where the complex envelope function $\tilde{W}_\ell$ given by Eqs. (16)-(16). Eq. (28) may further be transformed to establish its true dependency on time and wavenumber. To do this, we substitute the expression for the complex envelope function from Eq. (16). Thus, one obtains the following expression for the matrix $\tilde{\mathcal{P}}_{\ell i}$:

$$\tilde{\mathcal{P}}_{\ell i} \quad = \quad \frac{1}{\mathcal{I}_{0\ell}V_{0\ell}\Delta k_\ell\Delta\tau_i}\int_{t_i^-}^{t_i^+}\int_{k_\ell^-}^{k_\ell^+} dt\ dk\ \mathcal{F}(k)\mathcal{I}_0(k)V(k)\,e^{j\,\left(k\,x(t)+\phi(k)-\phi(k_\ell)\right)}. \tag{29}$$

Finally, we have defined everything that is needed to study Eq. (25), for the polychromatic fringe, which may equivalently be presented in a matrix form as below (where the complex matrix $\tilde{\mathcal{P}}_{\ell i}$ is given by Eq. (29)):

$$\begin{pmatrix} N_{\ell 1} \\ \dots \\ N_{\ell N} \end{pmatrix} = \begin{pmatrix} 1; & \mathsf{Im}\{\tilde{\mathcal{P}}_{\ell 1}\}; & \mathsf{Re}\{\tilde{\mathcal{P}}_{\ell 1}\} \\ \dots & \dots & \dots \\ 1; & \mathsf{Im}\{\tilde{\mathcal{P}}_{\ell N}\}; & \mathsf{Re}\{\tilde{\mathcal{P}}_{\ell N}\} \end{pmatrix} \begin{pmatrix} \mathcal{I}_{0\ell} \\ \mathcal{I}_{0\ell}V_{0\ell}\cos\phi(k_\ell) \\ \mathcal{I}_{0\ell}V_{0\ell}\sin\phi(k_\ell) \end{pmatrix}, \tag{30}$$

The obtained result Eq. (25), (29) (or, equivalently Eq. (30)) constitutes the general form of expression for the polychromatic fringe. We will use this result to finalize the development of the general from of the observational model for polychromatic case with arbitrary phase modulation. Ideally, one would need to determine not only three quantities $\mathcal{I}_{0\ell}V_{0\ell}\cos\phi(k_\ell)$, $\mathcal{I}_{0\ell}V_{0\ell}\sin(k_\ell)$, and $\mathcal{I}_{0\ell}$, but the full functional dependency of the original quantities. However, the finite width of the observational band-width Δk_ℓ complicates the estimation process by bringing the non-linearity in the observational equation via the envelop function W. Note that if one neglects the size of the bandwidth Δk_ℓ with respect to the mean wavenumber k_ℓ or $\Delta k_\ell/k_\ell \to 0$ (the envelop function becomes unity $W \to 1$), one recovers the full simplicity of the monochromatic case.[7]

3. SOLUTION FOR POLYCHROMATIC PHASORS WITH NOISY DATA

Currently in use, there are two fringe estimators, one for visibility (the unbiased estimator is V^2), and one for the phase (the unbiased estimator is the complex phasor). The V^2 estimator is already worked out in much detail (i.e. Refs.[2]-[10]) if the complex phasor estimator is completed. So the development of the complex phasor was the main purpose for the presented work. As it is known, the complex fringe visibility can be represented by a phasor; if the fringe is stable, we can add the phasors vectorially over multiple samples. This co-adding can provide an improved signal-to-noise ratio. To co-add the fringe phasors requires a phase reference, for instance the white light phase.

3.1. Parameterization of the Fringe Equation

In this Section we will develop optimally-weighted solution that accounts for a number of noise sources and will be applicable for a general case of delay modulation. For the purposes of clarity we will omit spectral index ℓ. All the obtained results are valid for any channel and thus could be easily reconstructed, if needed.

In the case of noisy data, observations of phonon-counts N_i are actually done with errors and, in reality, we observe $N_i = \bar{N}_i + \epsilon_i$, where $\bar{N}_i$ is the mean value of photon counts at the i-th temporal bin and ϵ_i is a random variable. We assume that ϵ_i are random variables that are primarily due to gaussian statistics. (This approach may be extended to incorporate other sources of noise. The corresponding results will be reported elsewhere.) that are distributed around zero and following relations are valid

$$N_i = \bar{N}_i + \epsilon_i, \qquad E(\epsilon_i) = 0, \qquad E(\epsilon_i^2) = \sigma_i^2. \tag{31}$$

In general one must account not only for the Gaussian statistics of read-out process, but also for the Poison statistic that governs photo-emission (and photo-counting) process. Thus, a correct approach would be to assume that ϵ_i is a sum of two terms $\epsilon_i = \mu\, \epsilon_i^G + (1-\mu)\, \epsilon_i^P$, where ϵ_i^G is Gaussian and ϵ_i^P is Poissonian variables and μ is a number between 0 and 1. Expected complication arises from the fact that standard deviation computed for the photon-counting Poissonian bias is actually proportional to the signal $\left(\sigma_i^P\right)^2 \propto \bar{\mathcal{I}}_i$ (see discussion in Ref.[10]). This issue is out of scope of the present paper and will be addressed at a later time.

We also assume that N_i are independent, therefore, we may form a diagonal covariance matrix for the quantities N_i (or equivalently for ϵ_i) with dispersions σ_i^2 on the diagonal:

$$C_y = \begin{pmatrix} \sigma_1^2; & 0; & ... & 0 \\ 0; & \sigma_2^2; & ... & 0 \\ ... & ... & ... & 0 \\ 0; & 0; & ... & \sigma_N^2 \end{pmatrix}, \qquad G_y = C_y^{-1} = \begin{pmatrix} \sigma_1^{-2}; & 0; & ... & 0 \\ 0; & \sigma_2^{-2}; & ... & 0 \\ ... & ... & ... & 0 \\ 0; & 0; & ... & \sigma_N^{-2} \end{pmatrix}, \tag{32}$$

where G_y is the matrix of weights. Therefore, in the case when noise is present in the data, equation (25) has following form:

$$\bar{N}_i + \epsilon_i = \mathcal{I}_{0\ell}\Big(1 + V_{0\ell}\sin\phi(k_\ell)\cdot \mathsf{Re}\{\tilde{\mathcal{P}}_{\ell i}\} + V_{0\ell}\cos\phi(k_\ell)\cdot \mathsf{Im}\{\tilde{\mathcal{P}}_{\ell i}\}\Big), \tag{33}$$

$$= \Big(1; \;\; \mathsf{Im}\{\tilde{\mathcal{P}}_{\ell i}\}; \;\; \mathsf{Re}\{\tilde{\mathcal{P}}_{\ell i}\}\Big) \begin{pmatrix} \mathcal{I}_{0\ell} \\ \mathcal{I}_{0\ell}V_{0\ell}\cos\phi(k_\ell) \\ \mathcal{I}_{0\ell}V_{0\ell}\sin\phi(k_\ell) \end{pmatrix} \tag{34}$$

or, equivalently,

$$\bar{N}_i + \epsilon_i = A_{i\alpha}X^\alpha, \tag{35}$$

with indexes i and α running as $i \in \{1, ..., N\}$ and $\alpha \in \{1,2,3\}$. Vector X^α is the vector to be determined and matrix $\mathbf{A}^T = A_{i\alpha}$ is the $3 \times N$ rotational matrix in the phase space. A maximum likelihood solution to the system of equations (35) may be given by the following system of equations

$$X^\alpha = \sum_i^N \mathbf{A}_\diamond^{\dagger\, i\alpha}\bar{N}_i, \qquad \text{where} \qquad \mathbf{A}_\diamond^\dagger = (\mathbf{A}^T G_y \mathbf{A})^{-1}\mathbf{A}^T G_y, \tag{36}$$

with $\mathbf{A}_\diamond^\dagger$ being an optimally-weighted pseudo-inverse matrix. Note that by choosing different gain matrix instead of optimally weighted least-squared matrix Eq. (32), one may obtain solution with different, specifically designed properties. Nevertheless, our solution has enough embedded generality as it allows for arbitrary properties of noise contribution, which will be further explored below.

3.2. Optimally-Weighted Pseudo-Inverse Matrix

In this Section we will find solution for the pseudo-inverse matrix $\mathbf{A}_\diamond^\dagger$. To construct this matrix we will use the weights matrix G_y given by Eq.(32) and matrix $\mathbf{A}_\diamond$ given by Eq.(28) as:

$$\mathbf{A}_i = \Big(1;\ \mathsf{Im}\{\tilde{\mathcal{P}}_i\};\ \mathsf{Re}\{\tilde{\mathcal{P}}_i\}\Big) \equiv \Big(1;\ s_i;\ c_i\Big). \tag{37}$$

where we denoted $s_i = \mathsf{Im}\{\tilde{\mathcal{P}}_i\} \equiv p_i \sin\pi_i, c_i = \mathsf{Re}\{\tilde{\mathcal{P}}_i\} \equiv p_i \cos\pi_i$. This parameterization is stemming from definition for the complex matrix, $\tilde{\mathcal{P}}_i$, given by Eq. (28) and presented in the form

$$\tilde{\mathcal{P}}_j = \mathsf{Re}\{\tilde{\mathcal{P}}_j\} + i\ \mathsf{Im}\{\tilde{\mathcal{P}}_j\} = p_j\, e^{i\pi_j}, \tag{38}$$

where p_j and π_j are given by $p_j = \sqrt{\mathsf{Re}^2\{\tilde{\mathcal{P}}_j\} + \mathsf{Im}^2\{\tilde{\mathcal{P}}_j\}}$, and $\pi_j = \mathsf{ArcTan}\Big\{\mathsf{Im}\{\tilde{\mathcal{P}}_j\}/\mathsf{Re}\{\tilde{\mathcal{P}}_j\}\Big\}$.

Calculation of $(\mathbf{A}^T G_y \mathbf{A})$ is straightforward even for the most general case of arbitrary number of temporal bins ($N \geq 3$) and with arbitrary integration intervals ($\Delta\tau_i \neq \Delta\tau_j$ for $i \neq j$) we find:

$$(\mathbf{A}^T G_y) = \begin{pmatrix} \frac{1}{\sigma_1^2}; & \frac{1}{\sigma_2^2}; & \cdots & \frac{1}{\sigma_N^2} \\ \frac{s_1}{\sigma_1^2}; & \frac{s_2}{\sigma_2^2}; & \cdots & \frac{s_N}{\sigma_N^2} \\ \frac{c_1}{\sigma_1^2}; & \frac{c_2}{\sigma_2^2}; & \cdots & \frac{c_N}{\sigma_N^2} \end{pmatrix}, \qquad (\mathbf{A}^T G_y \mathbf{A}) = \begin{pmatrix} \sum_i^N \frac{1}{\sigma_i^2} & \sum_i^N \frac{s_i}{\sigma_i^2} & \sum_i^N \frac{c_i}{\sigma_i^2} \\ \sum_i^N \frac{s_i}{\sigma_i^2} & \sum_i^N \frac{s_i^2}{\sigma_i^2} & \sum_i^N \frac{s_i c_i}{\sigma_i^2} \\ \sum_i^N \frac{c_i}{\sigma_i^2} & \sum_i^N \frac{s_i c_i}{\sigma_i^2} & \sum_i^N \frac{c_i^2}{\sigma_i^2} \end{pmatrix}. \tag{39}$$

By inverting the obtained result one constructs the covariance matrix Λ of the following structure:

$$\Lambda = (\mathbf{A}^T G_y \mathbf{A})^{-1} = \frac{1}{\Delta_\diamond} \begin{pmatrix} \frac{1}{2}\sum_{ij}^N \frac{(s_i c_j - c_i s_j)^2}{\sigma_i^2\sigma_j^2} & \frac{1}{2}\sum_{ij}^N \frac{(c_i - c_j)(s_i c_j - c_i s_j)}{\sigma_i^2\sigma_j^2} & -\frac{1}{2}\sum_{ij}^N \frac{(s_i - s_j)(s_i c_j - c_i s_j)}{\sigma_i^2\sigma_j^2} \\ \frac{1}{2}\sum_{ij}^N \frac{(c_i - c_j)(s_i c_j - c_i s_j)}{\sigma_i^2\sigma_j^2} & \frac{1}{2}\sum_{ij}^N \frac{(c_i - c_j)^2}{\sigma_i^2\sigma_j^2} & -\frac{1}{2}\sum_{ij}^N \frac{(s_i - s_j)(c_i - c_j)}{\sigma_i^2\sigma_j^2} \\ -\frac{1}{2}\sum_{ij}^N \frac{(s_i - s_j)(s_i c_j - c_i s_j)}{\sigma_i^2\sigma_j^2} & -\frac{1}{2}\sum_{ij}^N \frac{(s_i - s_j)(c_i - c_j)}{\sigma_i^2\sigma_j^2} & \frac{1}{2}\sum_{ij}^N \frac{(s_i - s_j)^2}{\sigma_i^2\sigma_j^2} \end{pmatrix},$$

where determinant of the matrix $(\mathbf{A}^T G_y \mathbf{A})$, $\Delta_\diamond = \det||(\mathbf{A}^T G_y \mathbf{A})||$, is given as

$$\Delta_\diamond = \frac{1}{2}\sum_{ijk}^N \frac{(s_i c_j - s_j c_i)}{\sigma_i^2\sigma_j^2\sigma_k^2}\Big[(s_i c_j - s_j c_i) + (s_j c_k - s_k c_j) + (s_k c_i - s_i c_k)\Big], \tag{40}$$

with summation over all temporal bins and running from 1 to N, namely $\forall\ \ \{i, j, k\} \in [1, N]$.

These intermediate results allow us to write the solution for the $(N \times 3)$ optimally-weighted pseudo-inverse matrix $\mathbf{A}_\diamond^\dagger = A_{\diamond k}^{\dagger\alpha}$ in the following compact form:

$$\mathbf{A}_\diamond^\dagger = (\mathbf{A}^T G_y \mathbf{A})^{-1}\mathbf{A}^T G_y = \frac{1}{\mathcal{D}^\diamond}\begin{pmatrix} \mathcal{A}_k^\diamond \\ \mathcal{B}_k^\diamond \\ \mathcal{C}_k^\diamond \end{pmatrix}, \tag{41}$$

where coefficients $\mathcal{A}^\diamond_k, \mathcal{B}^\diamond_k, \mathcal{C}^\diamond_k$ and $\mathcal{D}^\diamond$ depend on duration of each temporal bin, mean wavenumber and variances for the data taken in each bin, and are given by

$$
\begin{aligned}
\mathcal{A}^\diamond_k &= \sum_{ij}^{N} \frac{p_i p_j \sin[\pi_i - \pi_j]}{\sigma_i^2\sigma_j^2\sigma_k^2}\Big(p_i p_j \sin[\pi_i - \pi_j] + p_j p_k \sin[\pi_j - \pi_k] + p_k p_i \sin[\pi_k - \pi_i]\Big),\\
\mathcal{B}^\diamond_k &= \sum_{ij}^{N} \frac{(p_i \cos\pi_i - p_j \cos\pi_j)}{\sigma_i^2\sigma_j^2\sigma_k^2}\Big(p_i p_j \sin[\pi_i - \pi_j] + p_j p_k \sin[\pi_j - \pi_k] + p_k p_i \sin[\pi_k - \pi_i]\Big),\\
\mathcal{C}^\diamond_k &= -\sum_{ij}^{N} \frac{(p_i \sin\pi_i - p_j \sin\pi_j)}{\sigma_i^2\sigma_j^2\sigma_k^2}\Big(p_i p_j \sin[\pi_i - \pi_j] + p_j p_k \sin[\pi_j - \pi_k] + p_k p_i \sin[\pi_k - \pi_i]\Big),\\
\mathcal{D}^\diamond &= \sum_{k}^{N} \mathcal{A}^\diamond_k = \frac{1}{3}\sum_{ijk}^{N} \frac{1}{\sigma_i^2\sigma_j^2\sigma_k^2}\Big(p_i p_j \sin[\pi_i - \pi_j] + p_j p_k \sin[\pi_j - \pi_k] + p_k p_i \sin[\pi_k - \pi_i]\Big)^2. \qquad (42)
\end{aligned}
$$

3.3. Photon Noise-Optimized Solution for Polychromatic Phasors

An optimally-weighted solution for the quantities X^α may be obtained directly now from Eq.(36) with the help of expressions (41)-(42) in the following compact form:

$$
\mathcal{I}^\diamond_0 = \frac{1}{\mathcal{D}^\diamond}\sum_{k}^{N} \bar{N}_k\, \mathcal{A}^\diamond_k, \qquad \mathcal{I}^\diamond_0\, V^\diamond_0 \cos\bar{\phi}^\diamond = \frac{1}{\mathcal{D}^\diamond}\sum_{k}^{N} \bar{N}_k\, \mathcal{B}^\diamond_k, \qquad \mathcal{I}^\diamond_0\, V^\diamond_0 \sin\bar{\phi}^\diamond = \frac{1}{\mathcal{D}^\diamond}\sum_{k}^{N} \bar{N}_k\, \mathcal{C}^\diamond_k. \qquad (43)
$$

with coefficients of $\mathcal{A}^\diamond_k, \mathcal{B}^\diamond_k, \mathcal{C}^\diamond_k$ and $\mathcal{D}^\diamond$ given by Eqs. (42).

The obtained solution for the polychromatic visibility phasors given by Eq. (43) is given in the form of a linear combination of weighted photon counts recorded during a particular integration period. This form turned out to be very when analyzing contributions of CCD pixels that are systematically biased. The obtained result may be used to de-weight 'bad' pixels (in a statistical sense) and, thus, to reduce the problem of biases while estimating fringe parameters. This form allows to express an optimally-weighted solution for visibility, phase and the constant intensity terms in a familiar compact form:

$$
V^{\diamond 2}_0 = \frac{\big(\sum_k^N \bar{N}_k\, \mathcal{B}^\diamond_k\big)^2 + \big(\sum_k^N \bar{N}_k\, \mathcal{C}^\diamond_k\big)^2}{\big(\sum_k^N \bar{N}_k\, \mathcal{A}^\diamond_k\big)^2}, \qquad \bar{\phi}^\diamond = \mathrm{ArcTan}\Big[\frac{\sum_k^N \bar{N}_k\, \mathcal{C}^\diamond_k}{\sum_k^N \bar{N}_k\, \mathcal{B}^\diamond_k}\Big], \qquad \mathcal{I}^\diamond_0 = \frac{\sum_k^N \bar{N}_k\, \mathcal{A}^\diamond_k}{\sum_k^N \mathcal{A}^\diamond_k}. \qquad (44)
$$

The form of the obtained solution is simple to understand and it is straightforward to implement in the software codes. All the information necessary to calculate the $3N$ coefficients of $\mathcal{A}^\diamond_k, \mathcal{B}^\diamond_k, \mathcal{C}^\diamond_k$ and $\mathcal{D}^\diamond$ is presumed to be known before the experiment. Thus, for the case when $N = 8$ one would have to calculate only 24 numbers from Eq. (42). These numbers correspond to 8 numbers of $\mathcal{A}^\diamond_k$, 8 numbers of $\mathcal{B}^\diamond_k$ and 8 numbers of $\mathcal{C}^\diamond_k$. Then, by taking the data and estimating variances σ_i one may process the data with the help of Eqs. (43) or directly Eqs. (44). This approach is currently being utilized and corresponding results will be reported elsewhere.[11]

4. RECTANGULAR BANDPASS FILTER

To take advantage of the results derived in the previous section, we must first decide on the properties of the bandpass filter. This decision in return will affect the properties of the envelop function. Below we shall develop a model for a special case of the bandpass filter – a rectangular bandpass filter denoted here as $\mathcal{F}_\ell$, which is done analytically in the following form

$$
\mathcal{F}(k) = \sum_{\ell=1}^{L} \mathcal{F}_\ell(k), \qquad \text{where} \qquad \mathcal{F}_\ell(k) = \begin{cases} \mathcal{F}_{0\ell} = \text{const}, & k \in [k^-_\ell, k^+_\ell], \\ 0, & k \notin [k^-_\ell, k^+_\ell]. \end{cases} \qquad (45)
$$

We can also assume that the width of a spectral channel is small, so that both intensity of incoming radiation, $\mathcal{I}_0(k)$, and apparent visibility, $V(k)$, do not change within the spectral channel (in particular, this leads to $\hat{V}_{0\ell}(k) \equiv 1$ in Eqs. (9) and (10). Therefore, the following conditions are satisfied with a particular spectral channel, ℓ:

$$\mathcal{I}_0(k) = \text{const}, \qquad V(k) \;=\; \text{const}, \qquad \mathcal{F}_{0\ell} = \text{const}, \qquad \phi(k) - \phi(k_\ell) = d_{0\ell}(k - k_\ell) + \mathcal{O}(\frac{\partial^2\phi}{\partial k_\ell^2}), \tag{46}$$

where $d_{0\ell} = \frac{\partial\phi}{\partial k_\ell}$ is the delay within the ℓ-th channel.

One may perform integration of the fringe envelope function $\tilde{W}_\ell[x(t)]$ which is given by Eq. (16). To the second order in phase variation (i.e. $\mathcal{O}(\frac{\partial^2\phi}{\partial k_\ell^2})$), the resulted envelope function has following properties:

$$\tilde{W}_\ell\big[\Delta k_\ell, \phi(k_\ell), x_i\big] = \frac{1}{\Delta k_\ell}\int_{-\frac{1}{2}\Delta k_\ell}^{+\frac{1}{2}\Delta k_\ell} d\kappa\; e^{j\,\kappa\left(d_{0\ell}+x(t)\right)} = \frac{\sin[\frac{1}{2}\Delta k_\ell(d_{0\ell}+x(t))]}{\frac{1}{2}\Delta k_\ell(d_{0\ell}+x(t))}, \tag{47}$$

where we introduced a convenient variable, $\kappa = k - k_\ell$, and remember that $\Delta k_\ell = k_\ell^+ - k_\ell^-$ and $k_\ell = \frac{1}{2}(k_\ell^+ + k_\ell^-)$ are the width of the spectral channel and the mean wavenumber.

We can now present Eq. (28) for matrix $\tilde{\mathcal{P}}_{\ell i}$ as follows:

$$\tilde{\mathcal{P}}_{\ell i} \;=\; \frac{1}{\Delta\tau_i}\int_{t_i^-}^{t_i^+} dt\; e^{j\,k_\ell x(t)}\,\frac{\sin[\frac{1}{2}\Delta k_\ell(d_{0\ell}+x(t))]}{\frac{1}{2}\Delta k_\ell(d_{0\ell}+x(t))}. \tag{48}$$

At this moment, we introduce another convenient variable, $\tau = t - t_i$. Analogously, $\Delta\tau_i = t_i^+ - t_i^-$ and $t_i = \frac{1}{2}(t_i^+ + t_i^-)$ are the duration of the temporal integration within the i-th bin and the mean time for this bin correspondingly. This result is used to transform Eq. (48) as below:

$$\tilde{\mathcal{P}}_{\ell i} \;=\; e^{j\,k_\ell x(t_i)}\,\delta\tilde{\mathcal{P}}_{\ell i} \qquad \text{with} \quad \delta\tilde{\mathcal{P}}_{\ell i} = \frac{1}{\Delta\tau_i}\int_{-\frac{1}{2}\Delta\tau_i}^{+\frac{1}{2}\Delta\tau_i} d\tau\; e^{j\,k_\ell\left(x(t_i+\tau)-x(t_i)\right)}\,\frac{\sin[\frac{1}{2}\Delta k_\ell\big(d_{0\ell}+x(t_i+\tau)\big)]}{\frac{1}{2}\Delta k_\ell\big(d_{0\ell}+x(t_i+\tau)\big)}. \tag{49}$$

The obtained result explicitly depends on the functional form of the OPD modulation, $x(t)$. To integrate this equation one first needs to make certain assumptions on the temporal behavior of $x(t)$, which will be done in the following Sections. At this moment we present Eq. (25) in the following form:

$$N_{\ell i} \;=\; \mathcal{I}_{0\ell}\bigg(1 + V_{0\ell}\sin\big[\phi(k_\ell) + k_\ell x(t_i)\big]\cdot\mathsf{Re}\big\{\delta\tilde{\mathcal{P}}_{\ell i}\big\} + V_{0\ell}\cos\big[\phi(k_\ell) + k_\ell x(t_i)\big]\cdot\mathsf{Im}\big\{\delta\tilde{\mathcal{P}}_{\ell i}\big\}\bigg), \tag{50}$$

where the complex matrix $\delta\tilde{\mathcal{P}}_{\ell i}$ is given by Eq. (49). The importance of separating terms with $\delta\tilde{\mathcal{P}}_{\ell i}$ is that one can establish clear correspondence with monochromatic light, for which $\delta\tilde{\mathcal{P}}_{\ell i} = I_{\ell i}$, the identity matrix.

4.1. Stepping Phase Modulation

The stepping phase modulation realized when the pathlength difference is changes as a set of discrete values corresponding to a number of steps in the OPD space. mathematically this process represented as follows:

$$x(t) = \sum_{i=1}^{N=8} x(t_i), \qquad \text{where} \qquad x(t_i) = \begin{cases} x_i, & t \in [t_i^-, t_i^+], \\ 0, & t \notin [t_i^-, t_i^+], \end{cases} \tag{51}$$

with $t_i = \frac{1}{2}(t_i^+ + t_i^-)$. This procedure defines the temporal bins used to modulate the interferometric pattern.

Conditions (51) allow for a significant simplification of temporal integration in Eq. (49). It simply is leading to a substitution $x(t) \to x_i$ in Eq. (19), and matrix $\tilde{\mathcal{P}}_{\ell i}$ takes the following form

$$\tilde{\mathcal{P}}_{\ell i} = e^{j\,k_\ell x(t_i)} \frac{\sin[\frac{1}{2}\Delta k_\ell(d_{0\ell}+x_i)]}{\frac{1}{2}\Delta k_\ell(d_{0\ell}+x_i)}, \tag{52}$$

As a result, to the second order in the phase variation (i.e. $\phi(k_\ell) \approx \phi_\ell + \mathcal{O}(\frac{\partial^2\phi}{\partial k_\ell^2}\Big|_{k_\ell})$) Eq. (50) is taking the form:

$$N_{\ell i} = \mathcal{I}_{0\ell}\Big(1 + V_{0\ell}\cdot \mathrm{sinc}[\frac{1}{2}\Delta k_\ell(x_i+d_{0\ell})]\cdot \sin\big[\phi(k_\ell)+k_\ell x_i\big]\Big). \tag{53}$$

The obtained result Eq. (53) clearly depends on the particular form of the envelope function. As such, it has most of the parameters that are necessary for the phase estimation purposes in the case of wide bandwidth.

For the most practical cases the value of the sinc function will be close to sinc ~ 1. Indeed, let us analyze the argument of this function, $\frac{1}{2}\Delta k_\ell(x_i+d_{0\ell})$. Thus, one might expect that within the spectral channel the phase will stay constant, hence $d_{0\ell} = \partial\phi/\partial k_\ell \approx 0$. Furthermore, for the estimation purposes let us assume that all the step-sizes x_i are essentially $x_i = i\frac{\lambda_0}{N}$, where n is the total number of temporal integration bins, i is the number of a particular temporal bin, $i \in 1, N$, and λ_0 is the modulation wavelength or $\lambda_0 = \frac{2\pi}{k_0}$, where k_0 is the corresponding modulation wavenumber. Also remember that width of a spectral channel is related to the total SIM bandwidth as $\Delta k_\ell = \Delta k_{\mathsf{SIM}}/L$, where Δk_{SIM} is the total SIM bandwidth and L is the total number of spectral channels used for the white light fringe detection Therefore, one has

$$\frac{1}{2}\Delta k_\ell(x_i+d_{0\ell}) \approx \frac{1}{2}\Delta k_\ell x_i = \frac{\pi i}{LN}\frac{\Delta k_{\mathsf{SIM}}}{k_0}. \tag{54}$$

Assuming $\lambda^-_{\mathsf{SIM}} = 450$ nm and $\lambda^+_{\mathsf{SIM}} = 900$ nm, and $\lambda_0 = 900$ nm, thus yielding $\frac{\Delta k_{\mathsf{SIM}}}{k_0} = 1$. The maximal value for the expression (54) is realized when $i = N$, thus

$$\frac{\pi i}{LN}\frac{\Delta k_{\mathsf{SIM}}}{k_0} \le \frac{\pi}{L}. \tag{55}$$

Currently, there are different numbers of spectral channels used to process data from our testbeds. This number may be as large as $L = 80$ and as small as $L = 4$. Of coarse, when $L = 80$, the ratio π/L becomes $\pi/L = 0.03927$ and, thus, $\mathsf{sinc}[\frac{1}{2}\Delta k_\ell x_i]|_{L=80} = 0.99974$, and similarly for $L = 4$ the sinc function becomes $\mathsf{sinc}[\frac{1}{2}\Delta k_\ell x_i]|_{L=4} = 0.90032$. We will address the issue of phase estimation sensitivity to the width of a spectral channel Δk_ℓ at a later time.[11]

This observation allows us to present the sinc function as a series with respect to the small parameter $\Delta k_\ell z$ (with z being defined as $z = x_i + d_{0\ell}$) as

$$\mathrm{sinc}[\frac{\Delta k_\ell z}{2}] = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\Big[\frac{\Delta k_\ell z}{2}\Big]^{2n} = 1 - \frac{1}{3!}\Big[\frac{\Delta k_\ell z}{2}\Big]^2 + \frac{1}{5!}\Big[\frac{\Delta k_\ell z}{2}\Big]^4 + \mathcal{O}(\frac{1}{7!}\Big[\frac{\Delta k_\ell z}{2}\Big]^6), \tag{56}$$

one can present the expression Eq. (53) in the following form:

$$N_{\ell i} = \mathcal{I}_{0\ell}\Big[1 + V_{0\ell}\Big(1 - \frac{\Delta k_\ell^2(x_i+d_{0\ell})^2}{24} + \frac{\Delta k_\ell^4(x_i+d_{0\ell})^4}{1920}\Big)\sin\big[\phi(k_\ell)+k_\ell x_i\big]\Big]. \tag{57}$$

The obtained expression models the expected number of photons detected at the CCD for the rectangular bandpass filter and stepping phase modulation. It extends the results obtained for the monochromatic case on the finite size spectral bandwidth. This fact is indicated by the explicit dependency of the obtained result on the width of a spectral channel Δk_ℓ. (For the most of the interesting practical applications, the size of the delay within a particular spectral channel is very small $d_{0\ell} = \frac{\partial\phi}{\partial k_\ell} \approx 0$, which further simplifies Eq. (57)).

4.2. Ramping Phase Modulation

In this Section we will discuss another type of phase modulation – the case when the phase is linearly changes with time. This modulation utilizes the phase ramping technique.(For more details, see Refs.[3]-.[5]) To develop analytical solution we will be using the system equations developed above, specifically Eqs. (25) and (28).

The optical path difference for the case of ramping phase modulation is modeled as follows:

$$x(t) = x_0 + v \cdot t, \tag{58}$$

where x_0 is the initial PZT position and v is the instantaneous velocity of PZT motion. Remembering the definition for τ as $\tau = t - t_i$, and $\Delta\tau_i = t_i^+ - t_i^-$ and $t_i = \frac{1}{2}(t_i^+ + t_i^-)$, Eq. (49) takes the form:

$$\tilde{\mathcal{P}}_{\ell i} \;=\; e^{j\,k_\ell x(t_i)}\delta\tilde{\mathcal{P}}_{\ell i}, \qquad \text{with} \qquad \delta\tilde{\mathcal{P}}_{\ell i} = \frac{1}{\Delta\tau_i}\int_{-\frac{1}{2}\Delta\tau_i}^{+\frac{1}{2}\Delta\tau_i} d\tau\; e^{j\,k_\ell v\tau}\cdot\frac{\sin[\frac{1}{2}\Delta k_\ell z(\tau)]}{\frac{1}{2}\Delta k_\ell z(\tau)} \tag{59}$$

and $z(\tau) = d_{0\ell} + x(t_i) + v\,\tau$. This allows us to present Eq. (50) in the following form:

$$N_{\ell i} \;=\; \mathcal{I}_{0\ell}\Big(1 + V_{0\ell}\sin\left[\phi(k_\ell) + k_\ell x(t_i)\right]\cdot\mathsf{Re}\{\delta\tilde{\mathcal{P}}_{\ell i}\} + V_{0\ell}\cos\left[\phi(k_\ell) + k_\ell x(t_i)\right]\cdot\mathsf{Im}\{\delta\tilde{\mathcal{P}}_{\ell i}\}\Big), \tag{60}$$

where the complex matrix of additional rotation in the phase space, $\delta\tilde{\mathcal{P}}_{\ell i}$, is given by Eq. (49). Equation (60) may equivalently be presented in a matrix form as below

$$N_{\ell i} \;=\; \Big(1;\;\; \sin k_\ell x(t_i);\;\; \cos k_\ell x(t_i)\Big)\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathsf{Re}\{\delta\tilde{\mathcal{P}}_{\ell i}\} & -\mathsf{Im}\{\delta\tilde{\mathcal{P}}_{\ell i}\} \\ 0 & \mathsf{Im}\{\delta\tilde{\mathcal{P}}_{\ell i}\}; & \mathsf{Re}\{\delta\tilde{\mathcal{P}}_{\ell i}\}\end{pmatrix}\begin{pmatrix} \mathcal{I}_{0\ell} \\ \mathcal{I}_{0\ell}V_{0\ell}\cos\phi(k_\ell) \\ \mathcal{I}_{0\ell}V_{0\ell}\sin\phi(k_\ell)\end{pmatrix}. \tag{61}$$

The result of integration of Eq.(59) may not be presented in a compact analytical form. It rather could be expressed in the form of two functions defined as SinIntegral and CosIntegral. To simplify the analysis, the sinc function may be given in the form of power series expansion with respect to the small parameter $\Delta k_\ell z(\tau)$ as given by Eq. (56). This expansion allows us to present Eqs. (59) in the following form:

$$\delta\tilde{\mathcal{P}}_{\ell i} \;=\; \frac{1}{\Delta\tau_i}\int_{-\frac{1}{2}\Delta\tau_i}^{+\frac{1}{2}\Delta\tau_i} d\tau\; e^{j\,k_\ell v\tau}\cdot\Big(1 - \frac{\Delta k_\ell^2 z(\tau)^2}{24} + \frac{\Delta k_\ell^4 z(\tau)^4}{1920} + \mathcal{O}\big(\frac{\Delta k_\ell^6 z^6}{7!\cdot 2^6}\big)\Big), \tag{62}$$

where $z(\tau) = d_{0\ell} + x(t_i) + v\,\tau = z_i + v\,\tau$ with $z_i = d_{0\ell} + x(t_i)$. This equation, (62), was integrated to obtain coefficients $\mathsf{Re}\{\delta\tilde{\mathcal{P}}_{\ell i}\}$ and $\mathsf{Im}\{\delta\tilde{\mathcal{P}}_{\ell i}\}$ in the fringe equation Eq. (60), that may be written in the following form:

$$\mathsf{Re}\{\delta\tilde{\mathcal{P}}_{\ell i}\} \;=\; \frac{\sin[\frac{1}{2}k_\ell\,v\,\Delta\tau_i]}{\frac{1}{2}k_\ell\,v\,\Delta\tau_i}\Big[1 + \Big(2 - k_\ell^2 z_i^2 - (\frac{1}{2}k_\ell\,v\,\Delta\tau_i)^2\Big)\frac{\Delta k_\ell^2}{24k_\ell^2}\Big] - 2\cos[\frac{1}{2}k_\ell\,v\,\Delta\tau_i]\,\frac{\Delta k_\ell^2}{24k_\ell^2}, \tag{63}$$

$$\mathsf{Im}\{\delta\tilde{\mathcal{P}}_{\ell i}\} \;=\; 2k_\ell z_i\Big(-\frac{\sin[\frac{1}{2}k_\ell\,v\,\Delta\tau_i]}{\frac{1}{2}k_\ell\,v\,\Delta\tau_i} + \cos[\frac{1}{2}k_\ell\,v\,\Delta\tau_i]\Big)\frac{\Delta k_\ell^2}{24k_\ell^2}, \tag{64}$$

with $z_i = d_{0\ell} + x(t_i) \equiv d_{0\ell} + x_0 + v\,t_i$.

The obtained expression models the photon flux detected at the CCD for the case of rectangular bandpass filter and ramping phase modulation. It extends the results obtained for the monochromatic case on the finite size of spectral bandwidth. This fact is indicated by the explicit dependency of the obtained result on the width of a spectral channel Δk_ℓ.

5. DISCUSSION AND FUTURE PLANS

The main objective of this paper has been to introduce the reader to the concepts and the instrumental logic of the SIM astrometric observations, especially as they relate to estimation of the white light fringe parameters. The set of formulae described herein will serve as the kernel for the future mission analysis and simulations. We have also developed a set of expressions that may be used for fringe visibility and phase extraction for both SIM science and guide interferometers.

The logic of our method is straightforward: one first assumes the desirable properties of the bandpass filter, then finds the corresponding envelope function, and then applies the obtained expressions (which are valid for a generic case). The obtained solutions for the envelope function W and, most specifically, $\delta\tilde{\mathcal{P}}_{\ell i}$ may be directly substituted either in the expression for the complex visibility phasors Eq. (42) and (43), or into equations for the visibility, amplitude and phase of the fringe, given by Eqs.(44). We applied this formalism to the case of a rectangular bandpass. While the complex visibility phasors are linear with respect to photon counts, the explicit expressions for the fringe parameters are non-linear. This fact may be used to design specific properties of unbiased fringe estimators for processing the white light data. In our further work we will numerically address the problem of unbiased estimators for the fringe phase, visibility and group delay. Our simulations also show that, while the model of the rectangular bandpass filer is working quite, for the 'real life' one must account for the effect of leakage of light. This effect is concerned the leakage of light onto the studied spectrometer pixel of the detector from the adjacent pixels with different wavenumbers. At this moment, it seams more appropriate that a combination of the rectangular bandpass filter and the additional effect of light leakage from the adjacent pixels that must be included into the model of a CCD detector. The corresponding analysis, simulation results and implications for the instrument design will be reported elsewhere.[11]

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